PID Control

Proportional-Integral-Derivative (PID) controllers are one of the most commonly used types of controllers. They have numerous applications relating to temperature control, speed control, position control, etc. A PID controller provides a control signal that has a component proportional to the tracking error of a system, a component proportional to the accumulation of this error over time and a component proportional to the time rate of change of this error. This module will cover these different components and some of their different combinations that can be used for control purposes.

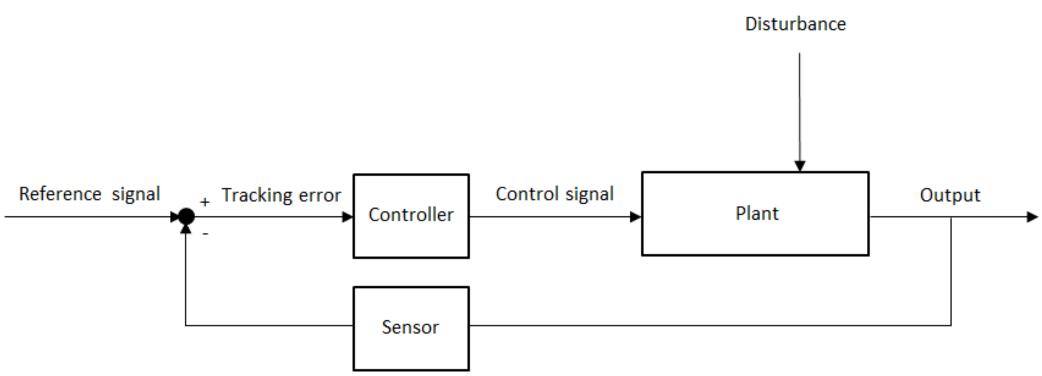


Fig. 1: System block diagram with feedback control

with(*inttrans*) :

(This command loads the functions required for computing Laplace and Inverse Laplace transforms. For more information on Laplace transforms, see the *Laplace Transforms and Transfer Functions* module.)

Proportional Control (P)

A proportional controller outputs a control signal $u_p(t)$ that is proportional to the error signal $e(t)$:

$$u_p(t) = K_p \cdot e(t)$$

... Eq. (1)

where K_p is the proportional gain. In the Laplace domain, this can be written as

$$U_p(s) = K_p \cdot E(s)$$

... Eq. (2)

First order systems with P control

The characteristic form of the transfer function of a first order plant is

$$G_p(s) = \frac{K}{\tau \cdot s + 1}$$

... Eq. (3)

where τ is the time constant and K is the DC Gain. With P control, the closed loop transfer function of the system is

$$G(s) = \frac{K_p \cdot K}{\tau \cdot s + 1 + K_p \cdot K}$$

... Eq. (4)

(This can be obtained using $G(s) = \frac{G_c(s) \cdot G_p(s)}{1 + G_c(s) \cdot G_p(s)}$ where $G_c(s)$ is the controller transfer function and $G_p(s)$ is the plant transfer function. See the *Block Diagrams, Feedback and Transient Response Specifications* module for more information.)

This transfer function is still a first order transfer function and can be written as

$$G(s) = \frac{\dfrac{K_p \cdot K}{1 + K_p \cdot K}}{\dfrac{\tau}{1 + K_p \cdot K} \cdot s + 1}$$

... Eq. (5)

Comparing Eq. (5) with Eq. (3), the closed loop time constant is

$$\tau_{closed\ loop} = \frac{\tau}{1 + K_p \cdot K}$$

... Eq. (6)

This shows that proportional control can be used to alter the rise time and settling time of a first order system.

Using Eq. (5) with a step input of magnitude R, the steady-state error for a first order system with proportional control is

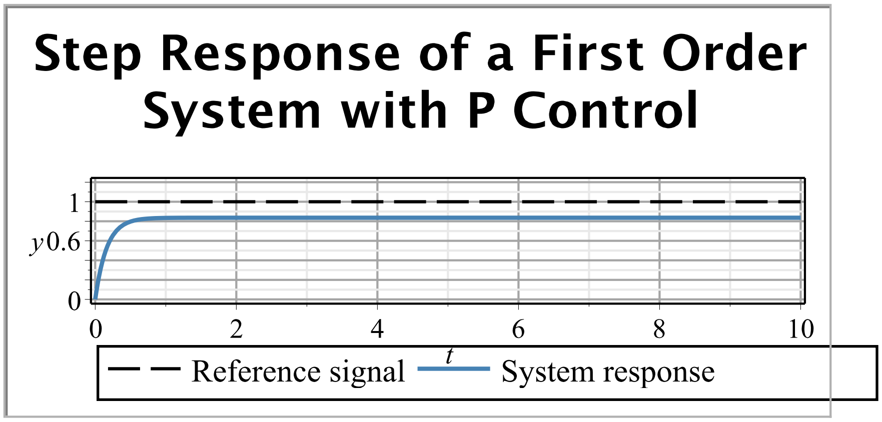
$$e_{ss} = \lim_{s \to 0} \left[s \cdot \left(\frac{R}{s} - \frac{R}{s} \cdot G(s) \right) \right] = R \cdot \frac{1}{1 + K_p \cdot K}$$

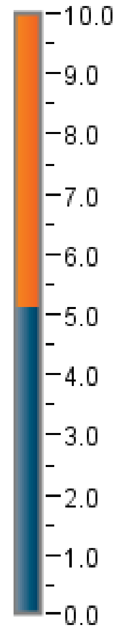
... Eq. (7)

This shows that the steady state error can be reduced by increasing the gain. However, to achieve zero steady-state error, the gain would have to approach infinity. Therefore, for a first order system, a proportional controller cannot be used to eliminate the step response steady state error.

The following plot shows the response of a system with a plant transfer function $G_p(s) = \frac{1}{s + 1}$ to a unit-step input.

	K_p





Rise time	Settling time	Steady-state error
$t_r =$ 0.361 sec	$t_s =$ 0.492 sec	$e_{ss} =$ 0.164

Second order systems with P control

The characteristic form of the transfer function of a second order plant is

$$G_p(s) = \frac{\omega_n^2}{s^2 + 2 \cdot \zeta \cdot \omega_n \cdot s + \omega_n^2}$$

... Eq. (8)

where ζ is the damping ratio and ω_n is the natural frequency. With P control, the closed loop transfer function of the system is

$$G(s) = \frac{K_p \cdot \omega_n^2}{s^2 + 2 \cdot \zeta \cdot \omega_n \cdot s + \left(1 + K_p\right) \cdot \omega_n^2}$$

... Eq. (9)

Comparing Eq. (9) with Eq. (8), the closed loop natural frequency and damping ratio are

$$\omega_{n,\ closed\ loop} = \omega_n \cdot \sqrt{1 + K_p}$$

... Eq. (10)

and

$$\zeta_{closed\ loop} = \frac{\zeta}{\sqrt{1 + K_p}}$$

... Eq. (11)

This shows that as K_p is increased, the natural frequency increases and the damping ratio decreases which results in larger and faster oscillations. Since the rise, settling and peak times all depend on both of these parameters it is possible to alter them by adjusting K_p. The same applies for the maximum overshoot which depends on the damping ratio.

With a step input of magnitude R, the steady-state error for the closed loop transfer function is

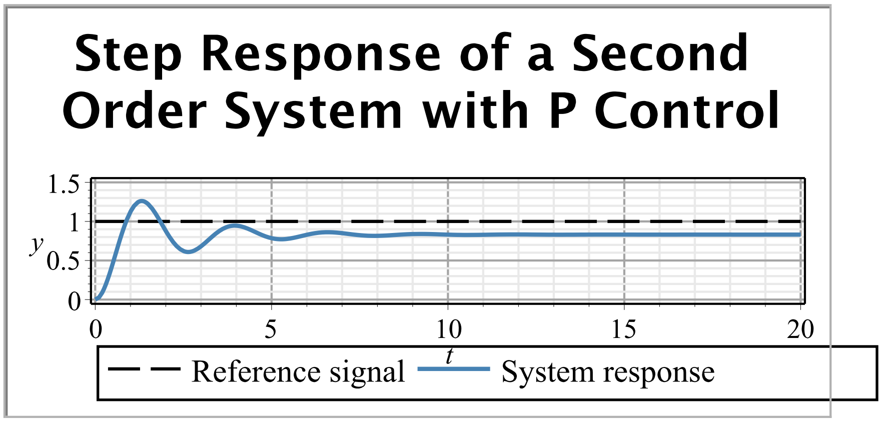
$$e_{ss} = \lim_{s \to 0}\left[s \cdot \left(\frac{R}{s} - \frac{R}{s} \cdot G(s)\right)\right] = R \cdot \frac{1}{1 + K_p}$$

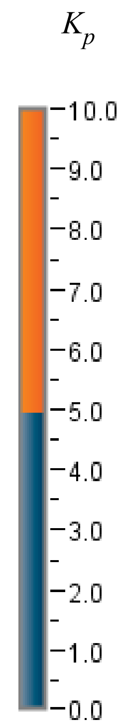
... Eq. (12)

This shows that the steady state error can be reduced by increasing the gain. However, similar to the proportional control of a first order system, zero steady-state error would require the proportional gain to approach infinity. Therefore, for a second order system, a proportional controller cannot be used to eliminate the step response steady state error.

The following plot shows the system response of a system with a plant transfer function

$G_p(s) = \dfrac{1}{s^2 + s + 1}$ to a unit-step input .





Damping ratio	Natural frequency	Damped frequency
$\zeta =$ 0.205	$\omega_n =$ 2.43 rad/s	$\omega_d =$ 2.38 rad/s
Rise time	**Maximum Overshoot**	**Steady-state error**
$t_r =$ 0.428 sec	$M_p =$ 51.9 %	$e_{ss} =$ 0.169

As can be seen, since all these parameters depend on K_p, the control on the system response specifications is limited. For example, this type of a controller does not allow both the steady-state error and the maximum overshoot to be reduced at the same time.

Integral Control (I)

An integral controller outputs a control signal $u_I(t)$ that is proportional to the integral of the error signal $e(t)$:

$$u_I(t) = K_I \cdot \int_0^t e(\tau)\, d\tau$$

... Eq. (13)

where K_I is the integral gain. In the Laplace domain, this can be written as

$$U_I(s) = \frac{K_I \cdot E(s)}{s}$$

... Eq. (14)

The integral component of a contoller provides a signal based on how long an error persists. It works to prevent this persistance of an error by increasing the control signal with time. This helps reduces the steady state error and in some cases, depending on the type of system and the type of reference signal, eliminates it. I control is usually not used on its own, however it is more effective than P control for eliminating the step response steady-state error of a first order plant. For a second order plant, using I control leads to a third order system that, depending on the system parameters, can result in unstable oscillations.

First order systems with I control

With I control, the closed loop transfer function of a first order system is

$$G(s) = \frac{\frac{K_I}{s} \cdot K}{\tau \cdot s + 1 + \frac{K_I}{s} \cdot K}$$

... Eq. (15)

This can be written as

$$G(s) = \frac{\frac{K_I \cdot K}{\tau}}{s^2 + \frac{s}{\tau} + \frac{K_I \cdot K}{\tau}}$$

... Eq. (16)

Therefore, the equivalent natural frequency and damping ratio are

$$\omega_n = \sqrt{\frac{K_I \cdot K}{\tau}}$$

... Eq. (17)

and

$$\zeta = \frac{1}{2 \cdot \sqrt{K_I \cdot K \cdot \tau}}$$

... Eq. (18)

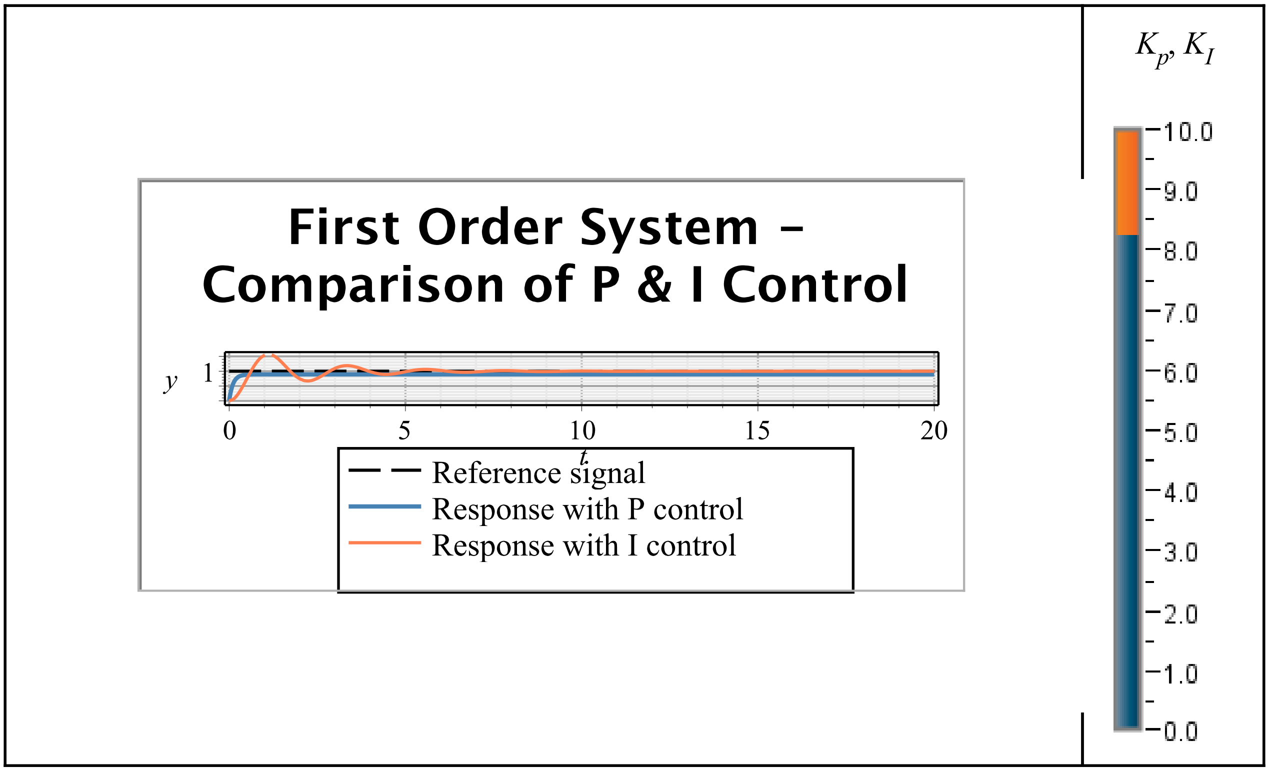
Also, with a step input of magnitude R, the steady-state error is

$$e_{ss} = \lim_{s \to 0}\left[s \cdot \left(\frac{R}{s} - \frac{R}{s} \cdot G(s) \right) \right] = 0$$

... Eq. (19)

This shows that integral control can be used to eliminate steady state error for a step input and have control over the response characteristics. Once again, similar to the case of P control for a second order system, since all the response specifications depend on the controller gain K_P, it is not possible to control them independently. For example, it is not possible to reduce the rise time and maximum overshoot simultaneously.

The following plot shows a comparison of the unit-step responses of a first order system with proportional control and with integral control (plant transfer function: $G_p(s) = \frac{1}{s+1}$).



Derivative Control (D)

A derivative controller outputs a control signal $u_d(t)$ that is proportional to the time derivative of the error signal $e(t)$:

$$u_d(t) = K_d \cdot \frac{d}{dt} e(t)$$

... Eq. (20)

where K_d is the derivative gain. In the Laplace domain, this can be written as

$$U_d(s) = s \cdot K_d \cdot E(s)$$

... Eq. (21)

The derivative component of a controller helps reduce overshoot. It is used to reduce the rate of change of the tracking error in order to prevent overshoot due to the inertia of the system. Derivative control is not covered in more detail by itself because it does not track error, only the rate of change of it.

Proportional Integral Control (PI)

PI control is a combination of proportional and integral control:

$$u(t) = K_p \cdot e(t) + K_I \cdot \int_0^t e(\tau)\, d\tau$$

... Eq. (22)

In the Laplace domain this can be written as

$$U(s) = \left(K_p + \frac{K_I}{s}\right) \cdot E(s)$$

... Eq. (23)

First order systems with PI control

With PI control, the closed loop transfer function of a first order system is

$$G(s) = \frac{\left(K_p + \frac{K_I}{s}\right) \cdot K}{\tau \cdot s + 1 + \left(K_p + \frac{K_I}{s}\right) \cdot K}$$

... Eq. (24)

This results in a second order system that can be written as

$$G(s) = \frac{\frac{(K_p \cdot s \cdot + K_I) \cdot K}{\tau}}{s^2 + \frac{1 + K \cdot K_p}{\tau} \cdot s + \frac{K_I \cdot K}{\tau}}$$

... Eq. (25)

so the equivalent natural frequency and damping ratio are

$$\omega_n = \sqrt{\frac{K_I \cdot K}{\tau}}$$

... Eq. (26)

and

$$\zeta = \frac{K \cdot K_p + 1}{2 \cdot \sqrt{K_I \cdot K \cdot \tau}}$$

... Eq. (27)

The steady state error for a step input of magnitude R is

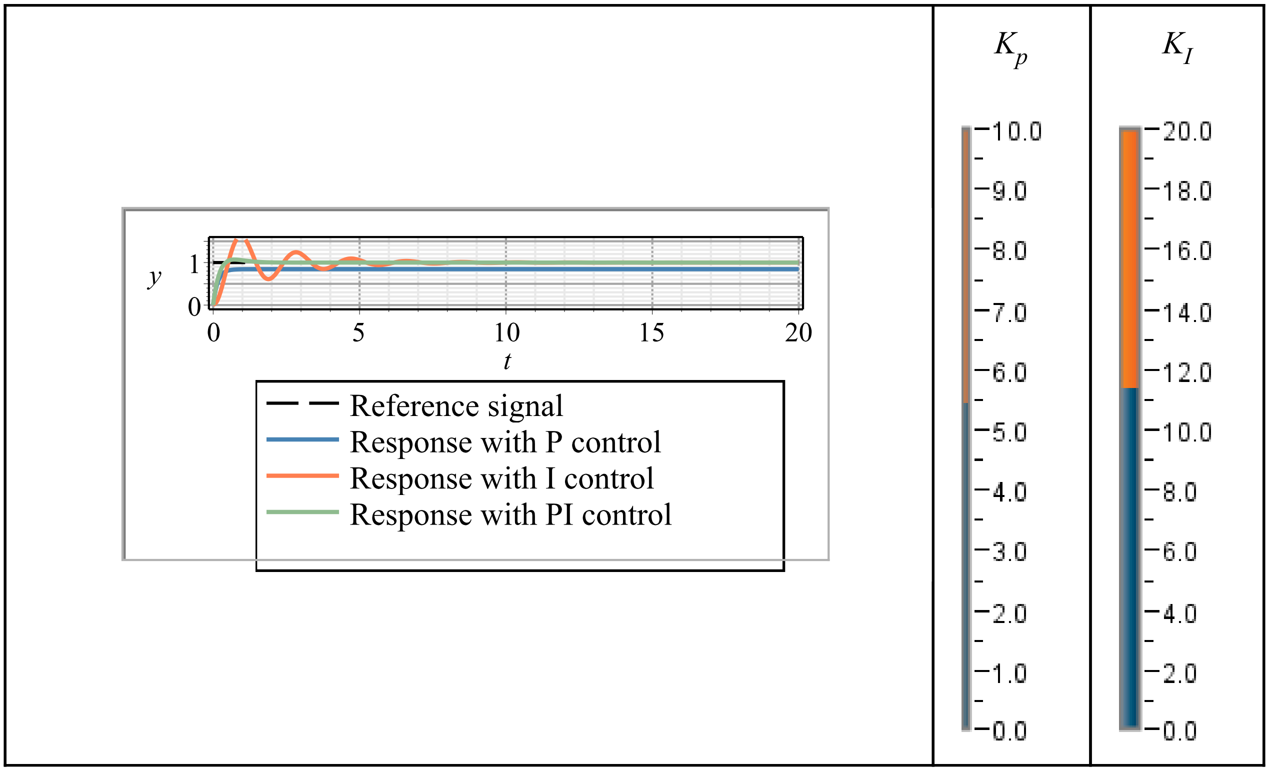
$$e_{ss} = \lim_{s \to 0} \left[s \cdot \left(\frac{R}{s} - \frac{R}{s} \cdot G(s) \right) \right] = 0$$

... Eq. (28)

This shows that proportional-integral control eliminates the step response steady state error and allows for more control over the transient response (compared to only P or only I control) because both the damping ratio and natural frequency can be altered using the gains. For example, it is now possible to reduce the rise time and maximum overshoot simultaneously.

The following plot shows a comparison of the unit-step responses of a first order system with proportional control, integral control and proportional-integral control (plant transfer function:

$G_p(s) = \frac{1}{s+1}$).



Second order system with PI control

With PI control, the closed loop transfer function of a second order system is

$$G(s) = \frac{(K_p \cdot s + K_I) \cdot \omega_n^2}{s^3 + 2 \cdot \zeta \cdot \omega_n \cdot s^2 + (1 + K_p) \cdot \omega_n^2 \cdot s + K_I \cdot \omega_n^2}$$

.. Eq. (29)

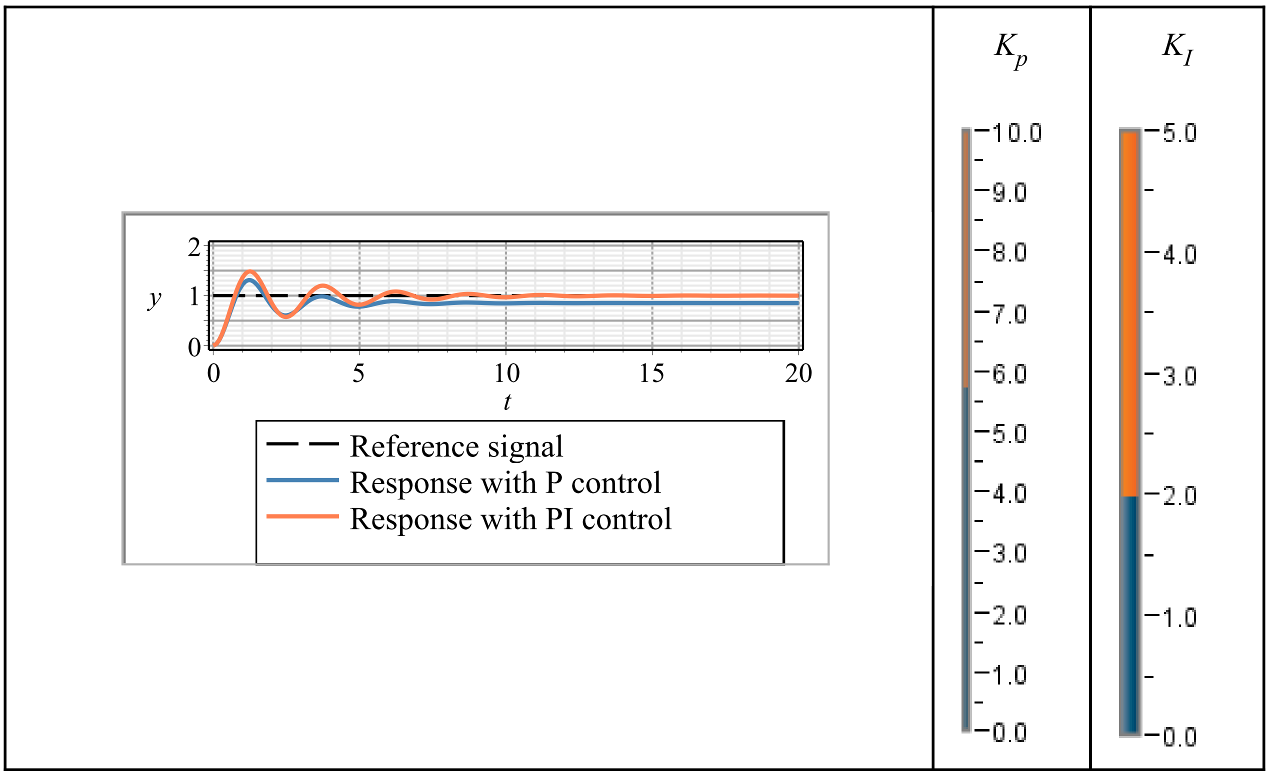
This is now a third order system with a zero. For a step input of magnitude R, the steady state error is

$$e_{ss} = \lim_{s \to 0} \left[s \cdot \left(\frac{R}{s} - \frac{R}{s} \cdot G(s) \right) \right] = 0$$

... Eq. (30)

This is an improvement over the other types of controllers discussed so far. Since the transfer function is a third order system and has a pole, it requires other methods like dominant pole analysis and root locus methods to analyze. It can also be tuned using approximate trial and error approaches to achieve the desired characteristics (use sliders below).

The following plot shows a comparison of the unit-step responses of a second order system with proportional control and proportional-integral control (plant transfer function: $G_p(s) = \frac{1}{s^2 + s + 1}$).



As can be observed, the control over the response is still limited.

Proportional Derivative Control (PD)

PD control is a combination of proportional and derivative control:

$$u(t) = K_p \cdot e(t) + K_d \cdot \frac{\mathrm{d}}{\mathrm{d}\,t} e(t)$$

... Eq. (31)

In the Laplace domain this can be written as

$$U(s) = \left(K_p + K_I \cdot s\right) \cdot E(s)$$

... Eq. (32)

First order systems with PD control

With PD control, the closed loop transfer function of a first order system is

$$G(s) = \frac{\left(K_p + K_d \cdot s\right) \cdot K}{\tau \cdot s + 1 + \left(K_p + K_d \cdot s\right) \cdot K}$$

... Eq. (33)

This results in a modified first order system with a zero that can be written as

$$G(s) = \frac{\left(K_p + K_d \cdot s\right) \cdot K}{\left(\tau + K \cdot K_d\right) \cdot s + 1 + K_p \cdot K}$$

... Eq. (34)

In this case, the steady state error for a step input remains the same as the steady state error with pure proportional control. There is no significant value added by including the derivative control.

Second order systems with PD control

With PD control, the closed loop transfer function of the system is

$$G(s) = \frac{\left(K_p + K_d \cdot s\right) \cdot \omega_n^2}{s^2 + \left(K_d \cdot \omega_n^2 + 2 \cdot \zeta \cdot \omega_n\right) \cdot s + \omega_n^2 \cdot \left(1 + K_p\right)}$$

... Eq. (35)

This is now a second order system with a zero. The step response steady state error is the same as with proportional control:

$$e_{ss} = \lim_{s \to 0} \left[s \cdot \left(\frac{R}{s} - \frac{R}{s} \cdot G(s) \right) \right] = \frac{1}{1 + K_p}$$

... Eq. (36)

For the design of such a system, the transfer function can be reduced to a second order system by ignoring the effect of the zero and calculating estimates for the required gains to meet the desired specifications. Then the response can be plotted and the gains can be tuned using a trial and error approach until the specifications are met. As can be seen from the transfer function, PD control allows for both the damping ratio and natural frequency to be controlled separately. For the approximate second order system, the natural frequency and damping ratio are

$$\omega_{n,\ closed\ loop} = \omega_n \cdot \sqrt{1 + K_p}$$

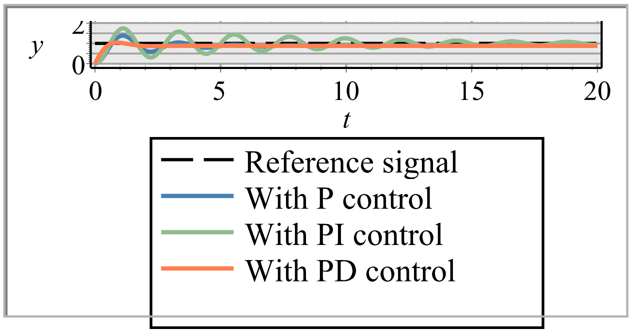
... Eq. (37)

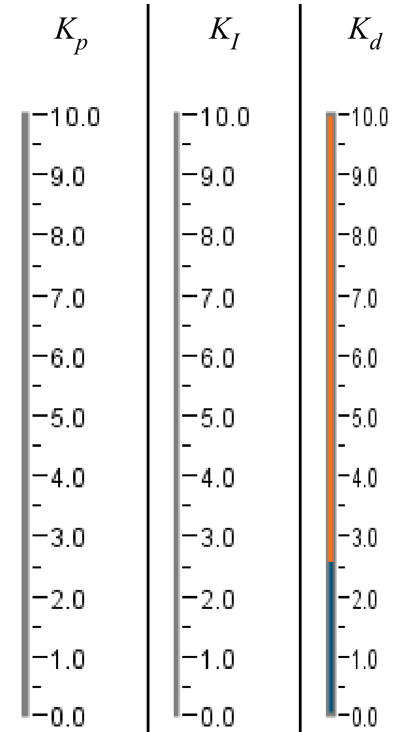
and

$$\zeta_{closed\ loop} = \frac{K_d \cdot \omega_n + 2 \cdot \zeta}{2 \cdot \sqrt{1 + K_p}}$$

... Eq. (38)

The following plot shows a comparison of the unit-step responses of a second order system with P control and PD control (plant transfer function: $G_p(s) = \frac{1}{s^2 + s + 1}$).





PID Control

PID control is a combination of proportional, integral and derivative control:

$$u(t) = K_p \cdot e(t) + K_I \cdot \int_0^t e(\tau)\, \mathrm{d}\tau + K_d \cdot \frac{\mathrm{d}}{\mathrm{d}t} e(t)$$

... Eq. (39)

In the Laplace domain this can be written as

$$U(s) = \left(K_p + \frac{K_I}{s} + s \cdot K_d\right) \cdot E(s)$$

... Eq. (40)

First order systems with PID

With PID control, the closed loop transfer function of a first order system is

$$G(s) = \frac{\left(K_p + \frac{K_I}{s} + K_d \cdot s\right) \cdot K}{\tau \cdot s + 1 + \left(K_p + \frac{K_I}{s} + K_d \cdot s\right) \cdot K}$$

... Eq. (41)

This results in a second order system with two zeros and can be written as

$$G(s) = \frac{\frac{\left(K_p \cdot s \cdot + K_I + K_d \cdot s^2\right) \cdot K}{\left(\tau + K_d \cdot K\right)}}{s^2 + \left(\frac{1 + K_p \cdot K}{\tau + K_d \cdot K}\right) \cdot s + \frac{K_I \cdot K}{\tau + K_d \cdot K}}$$

... Eq. (42)

The additional derivative term does not provide significant benefit over a PI controller and results in an increase in complexity.

Second order system with PID

With PID control, the closed loop transfer function for a second order system is

$$G(s) = \frac{\frac{\left(K_p \cdot s + K_I + K_d \cdot s^2\right) \cdot \omega_n^{\ 2}}{s\left(s^2 + 2 \cdot \zeta \cdot \omega_n \cdot s + \omega_n^{\ 2}\right)}}{1 + \frac{\left(K_p \cdot s + K_I + K_d \cdot s^2\right) \cdot \omega_n^{\ 2}}{s\left(s^2 + 2 \cdot \zeta \cdot \omega_n \cdot s + \omega_n^{\ 2}\right)}}$$

... Eq. (43)

or

$$G(s) = \frac{\left(K_p \cdot s + K_I + K_d \cdot s^2\right) \cdot \omega_n^{\ 2}}{s^3 + \left(2 \cdot \zeta \cdot \omega_n + K_d \cdot \omega_n^{\ 2}\right) \cdot s^2 + \omega_n^{\ 2} \cdot \left(1 + K_p\right) \cdot s + K_I \cdot \omega_n^{\ 2}}$$

... Eq. (44)

This is a third order system with two zeros. The three gains give complete control over the three poles of the system which means that this type of controller can be used to control the response characteristics better than the other types of controllers mentioned in this module.

With a step input of magnitude R, the steady-state error for the closed loop transfer function for is

$$e_{ss} = \lim_{s \to 0} \left[s \cdot \left(\frac{R}{s} - \frac{R}{s} \cdot G(s) \right) \right] = 0$$

... Eq. (45)

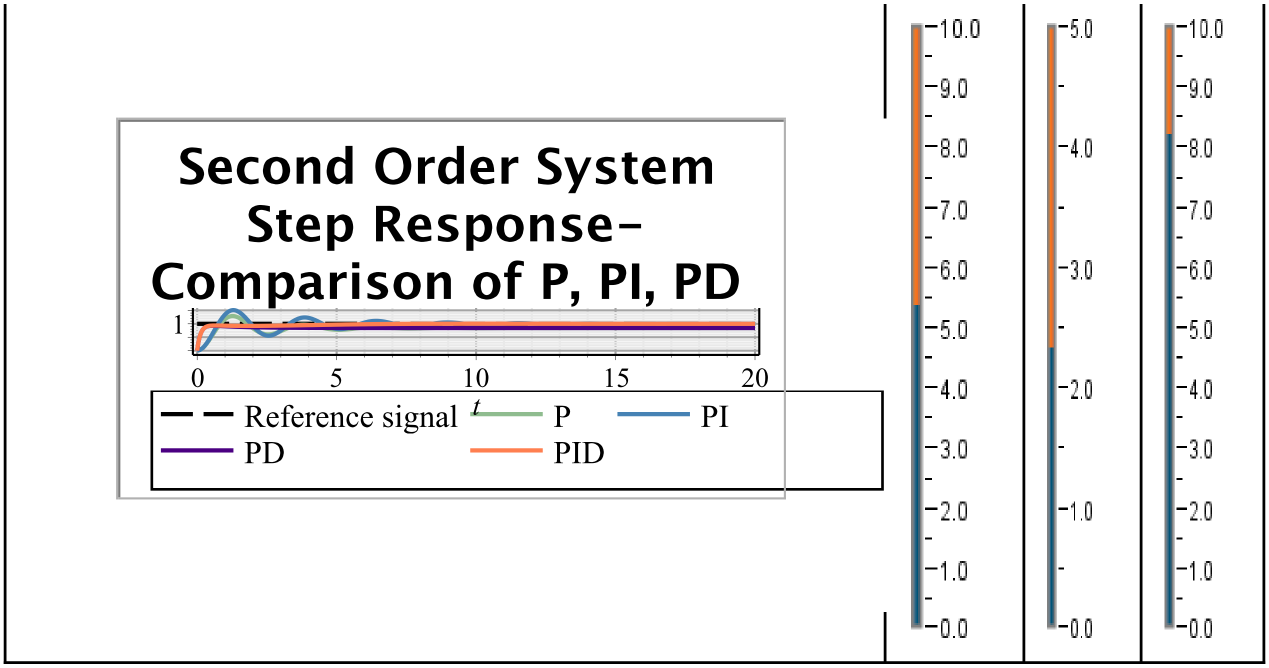
With a ramp input of solpe R, the steady-state error for the closed loop transfer function for is

$$e_{ss} = \lim_{s \to 0} \left[s \cdot \left(\frac{R}{s^2} - \frac{R}{s^2} \cdot G(s) \right) \right] = \frac{R}{K_I}$$

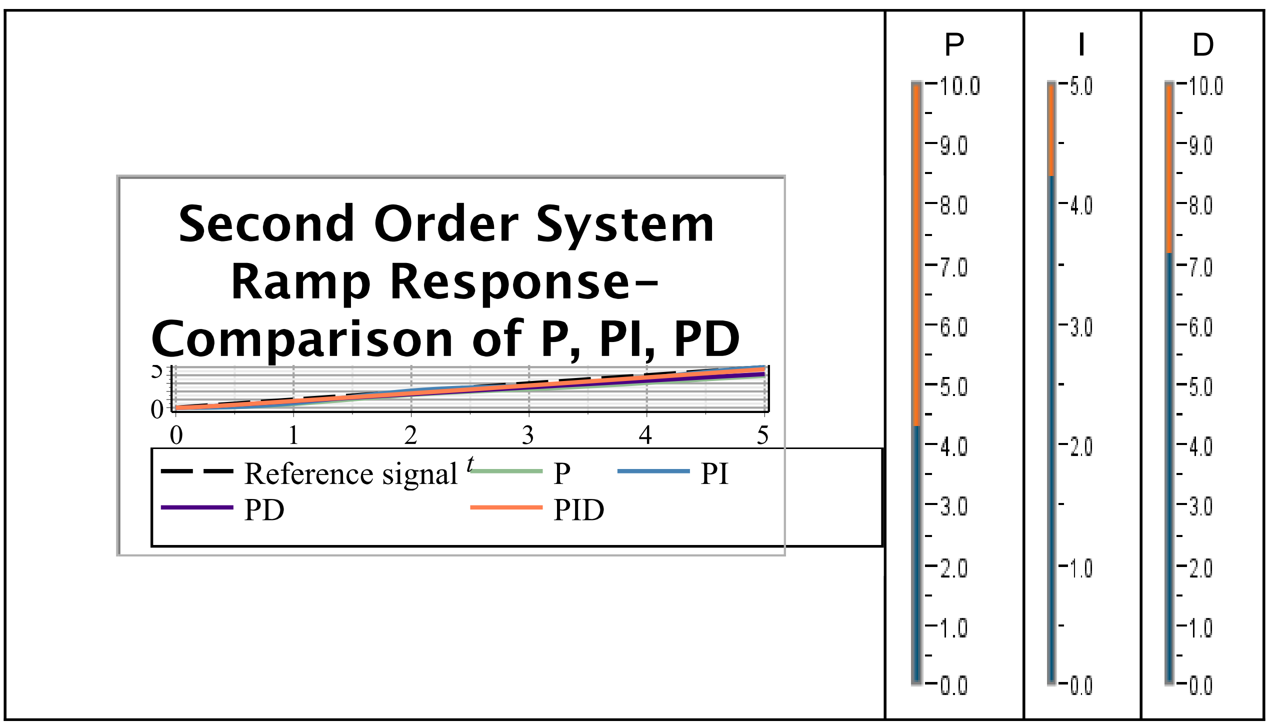
...Eq. (46)

The following plot shows a comparison of the unit-step response of a second order system with P, PD, PI and PID control (plant transfer function: $G_p(s) = \frac{1}{s^2 + s + 1}$).

	P	I	D



The following plot shows a comparison of the unit-ramp responses of a second order system with P, PD, PI and PID control (plant transfer function: $G_p(s) = \dfrac{1}{s^2 + s + 1}$).



Example 1: PID Motor speed control

Problem Description: The DC motor of a cooling fan has the following specifications:

Table 1: Motor specifications

Parameter	Value
Back-EMF constant, K	$85 \cdot 10^{-3}$ [V·s/rad]
Internal resistance, R	0.55 [Ω]
Internal inductance, L	$25 \cdot 10^{-3}$ [H]
Rotor moment of inertia, J	0.06 [kg·m^2]

The total moment of inertia of the fan blades is 0.03 kg·m^2 and the viscous resistance due to air resistance and bearing friction can be estimated as 0.05 N·m·s/rad. This system receives an input signal with the required rotational speed. Determine values of the controller gains of a controller that ensure that the maximum overshoot is less than 1%, the step response steady-state error is 0, the rise time is less than 0.1 seconds and the 1% settling time is less than 2 seconds.

Solution

restart :

Data:

$K := 85 \cdot 10^{-3}$: [V·s/rad]

$R := 0.55$: [Ω]

$L := 25 \cdot 10^{-3}$: [H]

$J := 0.06 + 0.03$: [kg·m^2]

$b := 0.05$: [N·m·s/rad]

The equivalent circuit of the motor consists of a voltage source, a resistor, an inductor and a "back EMF" voltage source:

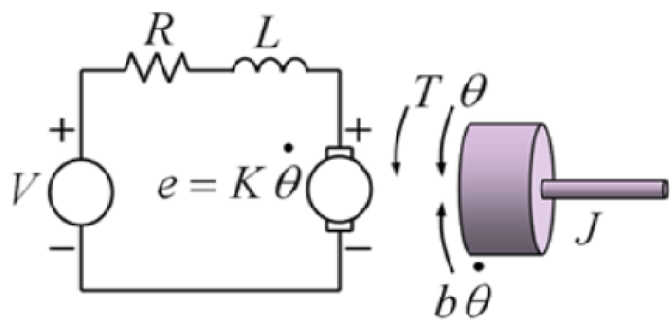


Fig. 2: DC Motor model

The back EMF depends on the rate of rotation and can be expresses as

$$e = K \cdot \omega(t)$$

where K is the back-EMF constant and $\omega(t)$ is the angular speed. The torque on the rotor is proportional to the armature current $i(t)$ and can be expresses as

$$T = K \cdot i(t)$$

The dynamic equation for the circuit is

$$V(t) = i(t) \cdot R + L \cdot \frac{d}{dt} i(t) + K \cdot \omega(t)$$

where $V(t)$ is the input voltage, R is the resistance of the resistor and L is the inductance of the inductor. The Laplace transform of this equation is

$$V(s) = (R + s \cdot L) \cdot I(s) + K \cdot \Omega(s)$$

... Eq. (47)

The dynamic equation for the rotor is

$$K \cdot i(t) = J \cdot \frac{d}{dt}\omega(t) + b \cdot \omega(t)$$

where J is the moment of inertia of the rotor and b is the damping constant. The Laplace transform of this equation is

$$K \cdot I(s) = J \cdot s \cdot \Omega(s) + b \cdot \Omega(s)$$

... Eq. (48)

Combining Eqs. (47) and (48) and eliminating $I(s)$ yields

$$V(s) = (R + s \cdot L) \cdot \left(\frac{J \cdot s \cdot \Omega(s) + b \cdot \Omega(s)}{K} \right) + K \cdot \Omega(s)$$

This equation can be rearranged to obtain the plant transfer function:

$$G_p = \frac{\Omega(s)}{V(s)} = \frac{K}{(J \cdot L) \cdot s^2 + (J \cdot R + L \cdot b) \cdot s + \left(R \cdot b + K^2\right)}$$

With the specifications given in the problem statement, the transfer function is

$$G_p := \frac{K}{(J \cdot L) \cdot s^2 + (J \cdot R + L \cdot b) \cdot s + \left(R \cdot b + K^2\right)}$$

$$\frac{17}{200\left(0.002250000000\, s^2 + 0.05075000000\, s + 0.03472500000\right)} \quad \textbf{(7.1.1)}$$

The next step is to determine the gains of a suitable controller. To be able to achieve a steady state error of zero for a step input, and be able to control both the rise time and maximum overshoot, the controller has to be a PID controller. The integral component is required to eliminate the steady-state error and the proportional and derivative terms give control over the rise time and the maximum overshoot. The controller transfer function is

$$G_c := K_p + \frac{K_I}{s} + K_d \cdot s :$$

and the closed loop transfer function is

$$G := \frac{G_p \cdot G_c}{1 + G_p \cdot G_c}$$

$$\frac{17}{200}\left(K_p + \frac{K_I}{s} + K_d s\right) \Bigg/ \left((0.002250000000\, s^2 + 0.05075000000\, s + 0.03472500000) \left(\frac{17}{200} \frac{K_p + \frac{K_I}{s} + K_d s}{0.002250000000\, s^2 + 0.05075000000\, s + 0.03472500000} + 1 \right) \right) \quad \textbf{(7.1.2)}$$

$collect(simplify(\textbf{(7.1.2)}), s)$

$$\frac{17.\left(K_d s^2 + K_p s + K_I\right)}{0.4500000000\, s^3 + (17.\, K_d + 10.15000000)\, s^2 + (17.\, K_p + 6.945000000)\, s + 17.\, K_I} \quad \textbf{(7.1.3)}$$

This is a third order system with two zeros. There are many different methods that are used for PID design. Here we will use a method that first involves calculating estimates for the controller gains by setting $K_I \approx 0$ and then using a trial and error approach to fine tune these gains to achieve the specifications. If we set $K_I \approx 0$ the closed loop system reduces to a second order system with a zero:

$$\frac{\frac{9.\,(K_d s + K_p)}{0.45}}{s^2 + \frac{(17.\, K_d + 10.15000000)}{0.45} s + \frac{(17.\, K_p + 6.945000000)}{0.45}}$$

Eq. (49)

If we ignore the effect of the zero, the damping ratio required for a maximum overshoot of 1% can be calculated:

$$\zeta_d := solve\left(0.01 = e^{-\frac{\zeta_d \cdot \pi}{\sqrt{1-\zeta_d^{\,2}}}}, \zeta_d\right)$$

$$0.8260850546 \tag{7.1.4}$$

Similarly the natural frequency required for a rise time 0.1 sec can be calculated:

$$\omega_n := solve\left(.1 = \frac{1 - 0.4167 \cdot \zeta_d + 2.917 \cdot \zeta_d^{\,2}}{\omega_n}, \omega_n\right)$$

$$26.46379339 \tag{7.1.5}$$

Using these values, the denominator of a second order transfer function that meets the overshoot and rise time specifications is:

$$s^2 + 2 \cdot \omega_n \cdot \zeta_d \cdot s + \omega_n^{\,2}$$

$$s^2 + 43.72268841\, s + 700.3323606 \tag{7.1.6}$$

Equating this to the denominator of Eq. (49), we get

$$s^2 + \frac{(9.\, K_d + 16.1875)}{0.53125}\, s + \frac{(9.\, K_p + 8.148)}{0.53125} = s^2 + 43.72268841\, s + 700.3323606 :$$

This equation can be used to solve for K_p and K_d:

$$K_d := solve\left(\frac{(17.\, K_d + 10.15000000)}{0.45} = 43.722, K_d\right)$$

$$0.5602882356 \tag{7.1.7}$$

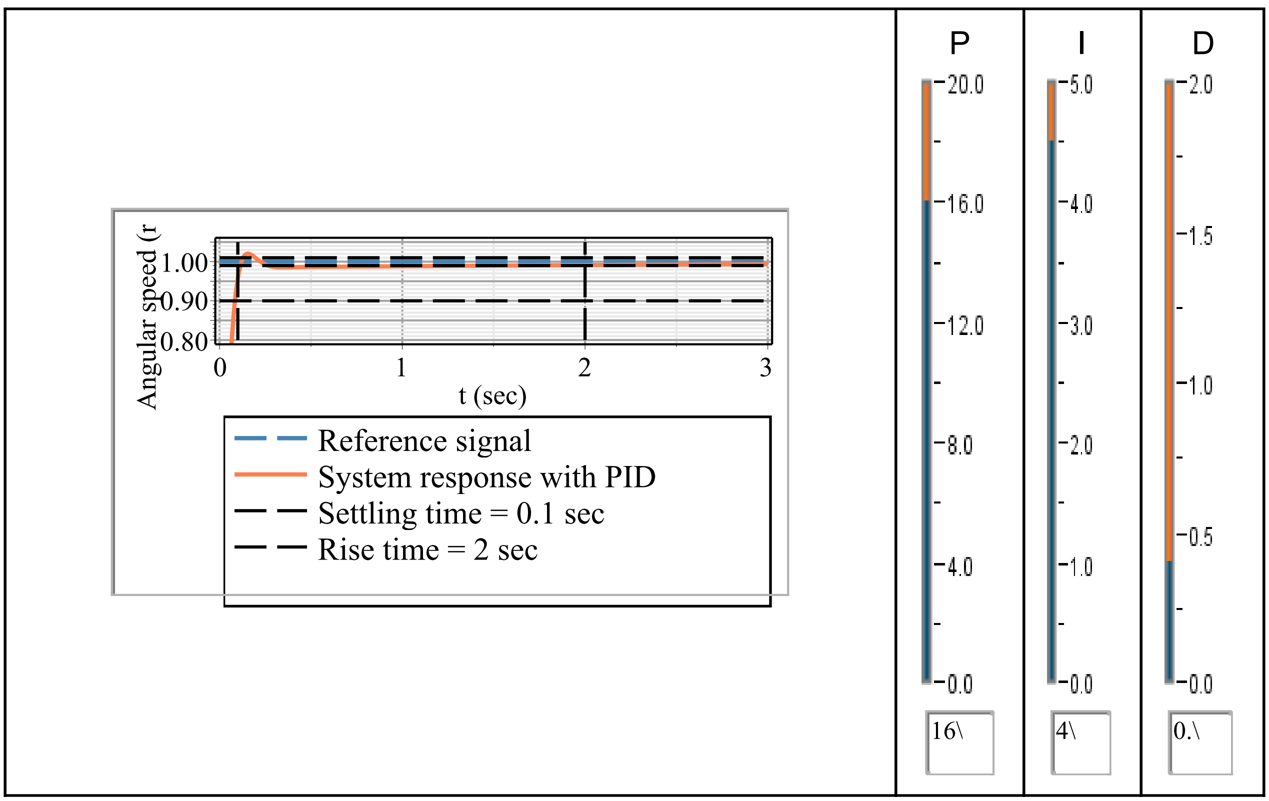
and

$$K_p := solve\left(\frac{(17.\,K_p + 6.945000000)}{0.45} = 700.3323606, K_p\right)$$

$$18.12968014 \quad \textbf{(7.1.8)}$$

These values are the estimates that can be used as a starting point to obtain the required controller gains using an iterative approach. The integral term eliminates the steady-state error for a step input and it's magnitude determines the settling time. Therfore the value of the integral gain can be increased until the settling time specification is met and then the other gains can be adjusted to ensure that the remaining specifications are also met. Using the gauges below, it can be found that the specifications are met if $K_p \approx 16$, $K_I \approx 4.5$, and $K_d \approx 0.47$.

$with(plots):$



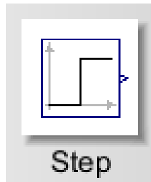
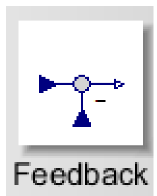
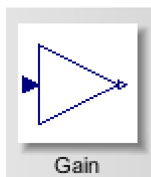
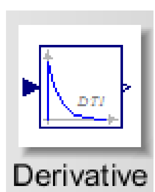
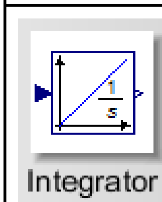
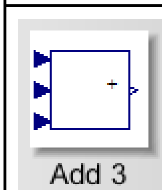
With MapleSim

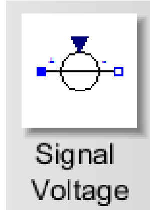
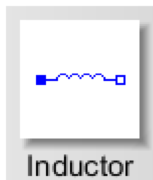
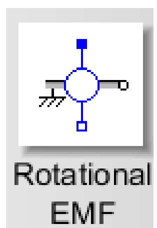
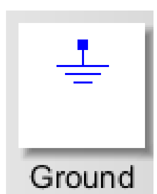
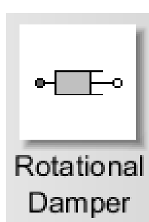
Constructing the model

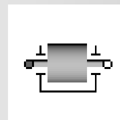
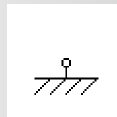
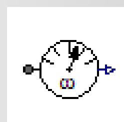
Step 1: Insert Components

Drag the following components into the workspace:

Table 2: Components and locations

Component	Location
	Signal Blocks > Common
	Signal Blocks > Common
	Signal Blocks > Common
	Signal Blocks > Mathematical > Functions
	Signal Blocks > Common
	Signal Blocks > Mathematical > Operators

	Electrical > Analog > Sources > Voltage
	Electrical > Analog > Common
	Electrical > Analog > Common
	Electrical > Analog > Common
	Electrical > Analog > Common
	1-D Mechanical > Rotational > Common
	1-D Mechanical >

 Inertia	Rotational > Common
 Rotational Fixed	1-D Mechanical > Rotational > Common
 Angular Velocity Sensor	1-D Mechanical > Rotational > Sensors

Step 2: Connect the components

Connect the components as shown in the following diagram (the dashed boxes are not part of the model, they have been drawn on top to help make it clear what the different components are for):

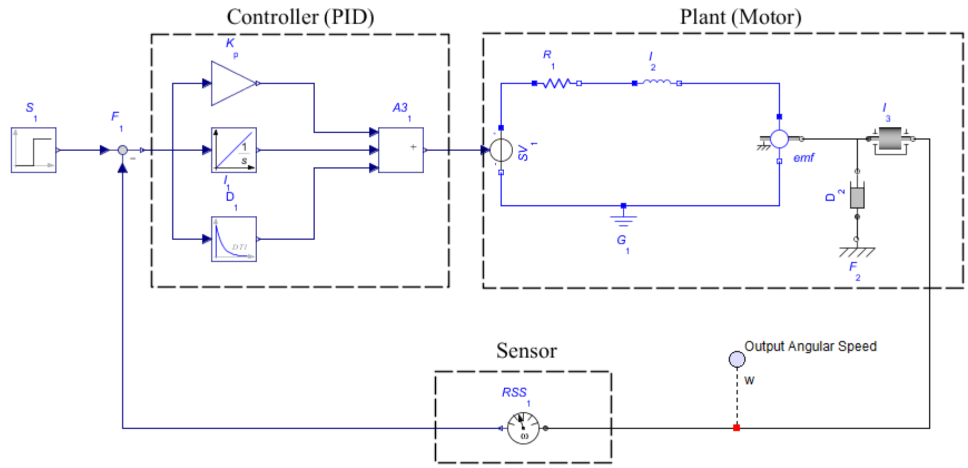


Fig. 3: Component diagram

Step 3: Set up the controller

1. Click the **Gain** component and enter the estimated K_p value calculated in the previous sub-section for the gain value **(*k*)**.
2. Click the **Derivative** component and enter the estimated K_d value calculated in the previous sub-section for the gain **(*k*)**.

Step 4: Set up the plant

1. Click the **Resistor** component and enter **0.55** Ω for the resistance **(*R*)**.
2. Click the **Integrator** component and enter **0.025** H for the inductance **(*L*)**.
3. Click the **Rotational EMF** component and enter **0.085** $\frac{N \cdot m}{A}$ for the **Transformation Coefficient (*k*)**.
4. Click the **Inertia** component and enter **0.09** $kg \cdot m^2$ for the moment of inertia **(*J*)**.
5. Click the **Rotational Damper** and enter **0.05** $\frac{N \cdot m \cdot s}{rad}$ for the **Damping constant (*d*)**.

Step 5: Run the simulation

1. Attach a **Probe** as shown in the diagram.
2. Click the probe and select **Speed** in the **Inspector** tab.

3. Click **Run Simulation** ( ).

4. Use a systematic trial and error approach to determine a combination of controller gains that satisfy the requirements.

To further study the behavior of the system, the input can be changed from a step input to one of the various other inputs (sinusoidal, pulse, ramp, etc.) available under **Sources** in the component library.

For example, the following plot shows the system response to an offset sinusoidal input with the gains set such that they meet the problem specifications.

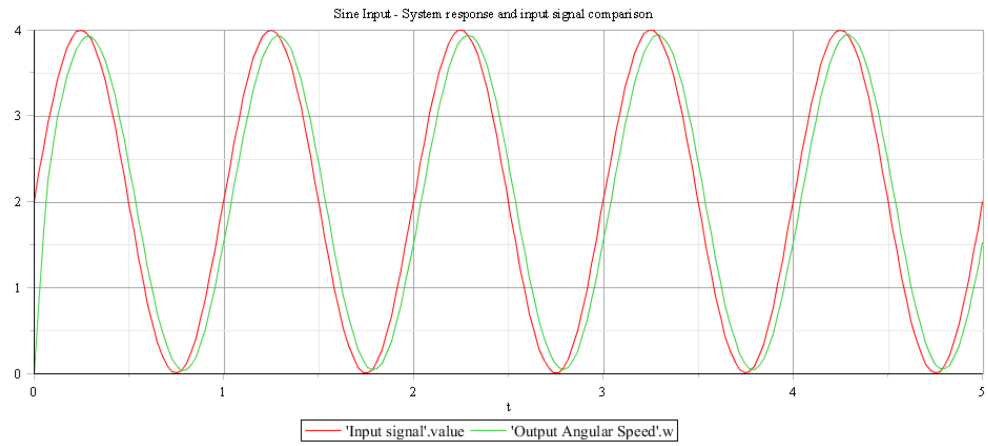


Fig 4: MapleSim plot of the response to an offset sinusoidal input - Angular speed (rad/s) vs. time (sec)

Also, if a CAD model is available, it can be used for visualization as shown below.

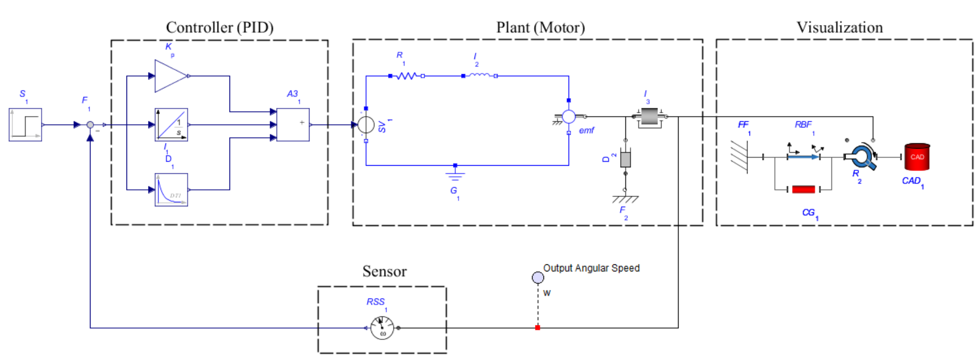
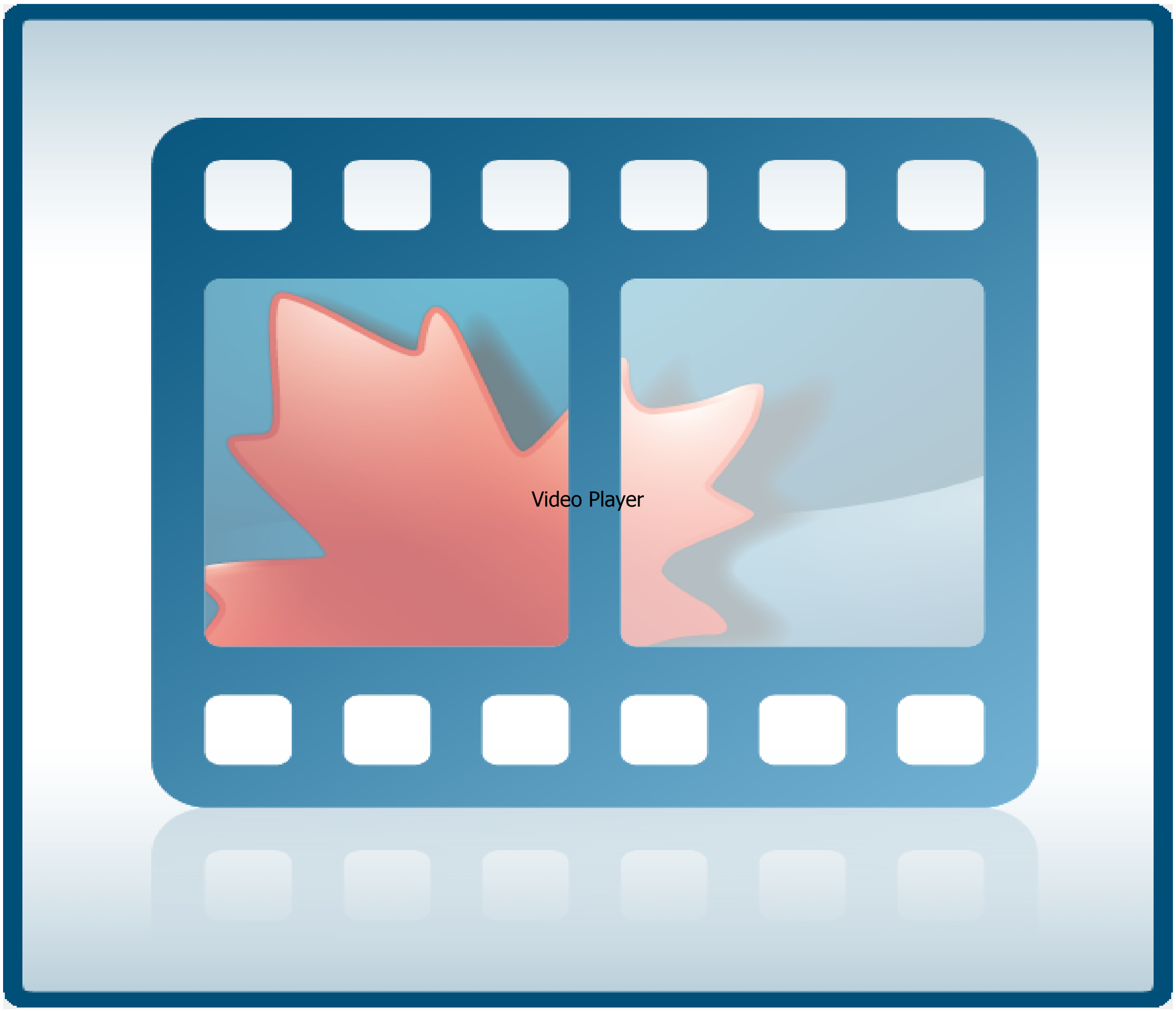


Fig. 5: Component diagram including a CAD attachment

The following video shows the system response for an offset sinusoidal input with a CAD model of the fan blades attached.



Video 1: MapleSim visualization of the response to an offset sinusoidal input.

Reference:
G.F. Franklin et al. "Feedback Control of Dynamic Systems", 5th Edition. Upper Saddle River, NJ, 2006, Pearson Education, Inc.